Model Efek Random Satu Arah

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1. Model Efek Random Satu Arah

Model : $y_{ij} = \mu + a_i + \varepsilon_{ij}, i = 1, 2, ..., I; j = 1, 2, ..., J$ (1.1)

Dalam model (1.1) $y_{ij}, a_i,$ dan ε_{ij} merupakan variabel random dengan asumsi

a. $\varepsilon_{ij} \sim N\left(0, \sigma^2\right) \Rightarrow E\left(\varepsilon_{ij}\right) = 0,\ E\left(\varepsilon_{ij}^2\right) = \mathrm{var}\left(\varepsilon_{ij}\right) = \sigma^2$ untuk semua i, j ,

b. $E\left(\varepsilon_{ij}, \varepsilon_{rs}\right) = \mathrm{cov}\left(\varepsilon_{ij}, \varepsilon_{rs}\right) = 0$ untuk semua $i \neq r$ atau $j \neq s$,

c. $a_i \sim N\left(0, \sigma^2\right) \Rightarrow E\left(a_i\right) = 0,\ E\left(a_i^2\right) = \mathrm{var}\left(a_i\right) = \sigma_a^2$ untuk semua i ,

d. $E\left(a_i a_j\right) = \mathrm{cov}\left(a_i, a_j\right) = 0$ untuk semua $i \neq j$,

e. $E\left(a_i \varepsilon_{ij}\right) = \mathrm{cov}\left(a_i, \varepsilon_{ij}\right) = 0$ untuk semua $i \neq j$,

Akibatnya,

f. $E\left(y_{ij}\right) = E\left(\mu + a_i + \varepsilon_{ij}\right) = \mu$,

g. $\mathrm{var}\left(y_{ij}\right) = \mathrm{var}\left(\mu + a_i + \varepsilon_{ij}\right) = \sigma_a^2 + \sigma^2$ untuk semua i, j ,

h. $\mathrm{cov}\left(y_{ij}, y_{ir}\right) = \sigma_a^2$ untuk semua $j \neq r$,

i. $\mathrm{cov}\left(y_{ij}, y_{rs}\right) = 0$ untuk semua $i \neq r$ atau $j \neq s$.

1.1 Estimasi Parameter

Estimasi parameter model efek random sama seperti pada model efek tetap. Oleh karena itu estimasi parameter-parameter dalam model efek random sebagai berikut:

a. $\hat{\mu} = \bar{y}_{..} = \dfrac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij}$, (1.2)

b. $\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} = \bar{y}_{i.} - \dfrac{1}{I} \sum_{i=1}^{I} \bar{y}_{i.}$, (1.3)

c. $SSA = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\overline{y}_{i.} - \overline{y}_{..}\right)^2 = J\sum_{i=1}^{I}\left(\overline{y}_{i.} - \overline{y}_{..}\right)^2$, (1.4)

d. $SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \overline{y}_{i.}\right)^2$, (1.5)

e. $SST = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \overline{y}_{..}\right)^2$. (1.6)

1.2 Expected Mean Squares

Mean Squares didefinisikan sebagai MSA = SSA/(I-1) dan MSE = SSE/(IJ-I). Untuk mendapatkan Expected Mean Squares E(MSA) dan E(MSE), pertama modifikasi (1.2) dan (1.3) sehingga menjadi:

$$\overline{y}_{i.} = \frac{1}{J}\sum_{j=1}^{J} y_{ij} = \frac{1}{J}\sum_{j=1}^{J}\left(\mu + a_i + \varepsilon_{ij}\right) = \mu + a_i + \frac{1}{J}\sum_{j=1}^{J}\varepsilon_{ij} = \mu + a_i + \overline{\varepsilon}_{i.}$$

dan

$$\overline{y}_{..} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\mu + a_i + \varepsilon_{ij}\right) = \mu + \overline{a}_{.} + \overline{\varepsilon}_{..}$$

$$E\left(SSA\right) = E\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\overline{y}_{i.} - \overline{y}_{..}\right)^2\right) = E\left(J\sum_{i=1}^{I}\left(\overline{y}_{i.} - \overline{y}_{..}\right)^2\right)$$

$$\Leftrightarrow E\left(J\sum_{i=1}^{I}\left(\mu + a_i + \overline{\varepsilon}_{i.} - \left(\mu + \overline{a}_{.} + \overline{\varepsilon}_{..}\right)\right)^2\right) = E\left(J\sum_{i=1}^{I}\left(a_i + \overline{\varepsilon}_{i.} - \overline{a}_{.} - \overline{\varepsilon}_{..}\right)^2\right)$$

$$\Leftrightarrow J\sum_{i=1}^{I} E\left(a_i^2 + \overline{\varepsilon}_{i.}^2 + \overline{a}_{.}^2 + \overline{\varepsilon}_{..}^2 - 2a_i\overline{a}_{.} - 2\overline{\varepsilon}_{i.}\overline{\varepsilon}_{..}\right)$$

$$\Leftrightarrow J\sum_{i=1}^{I}\left(E\left(a_i^2\right) + E\left(\overline{\varepsilon}_{i.}^2\right) + E\left(\overline{a}_{.}^2\right) + E\left(\overline{\varepsilon}_{..}^2\right) - 2E\left(a_i\overline{a}\right)_{.} - 2E\left(\overline{\varepsilon}_{i.}\overline{\varepsilon}_{..}\right)\right)$$

$$\Leftrightarrow J\sum_{i=1}^{I}\left(\sigma_a^2 + \frac{\sigma^2}{J} + \frac{\sigma_a^2}{I} + \frac{\sigma^2}{IJ} - 2\frac{\sigma_a^2}{I} - 2\frac{\sigma^2}{IJ}\right)$$

$$\Leftrightarrow IJ\left(\sigma_a^2 - \frac{\sigma_a^2}{I} + \frac{\sigma^2}{J} - \frac{\sigma^2}{IJ}\right) = \left(IJ - J\right)\sigma_a^2 + \left(I-1\right)\sigma^2 = \left(I-1\right)\left(\sigma^2 + J\sigma_a^2\right)$$

Sehingga: $E\left(MSA\right) = E\left(\frac{SSA}{I-1}\right) = J\sigma_a^2 + \sigma^2$.

Dengan cara serupa,

$$E\left(SSE\right)=E\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\overline{y}_{i.}\right)^2\right)=E\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\mu+a_i+\varepsilon_{ij}-\left(\mu+a_i+\overline{\varepsilon}_{i.}\right)\right)^2\right)$$

$$\Leftrightarrow E\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\varepsilon_{ij}-\overline{\varepsilon}_{i.}\right)^2\right)=E\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\varepsilon_{ij}^2+\overline{\varepsilon}_{i.}^2-2\varepsilon_{ij}\overline{\varepsilon}_{i.}\right)\right)$$

$$\Leftrightarrow \sum_{i=1}^{I}\sum_{j=1}^{J}\left(E\left(\varepsilon_{ij}^2\right)+E\left(\overline{\varepsilon}_{i.}^2\right)-2E\left(\varepsilon_{ij}\overline{\varepsilon}_{i.}\right)\right)=\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\sigma^2+\frac{\sigma^2}{J}-2\frac{\sigma^2}{J}\right)$$

$$\Leftrightarrow \left(IJ-I\right)\sigma^2=I\left(J-1\right)\sigma^2$$

Sehingga, $E\left(MSE\right)=E\left(\dfrac{SSE}{IJ-I}\right)=\sigma^2$.

1.3 Inferensi

Untuk menguji perbedaan efek variabel random digunakan uji dengan hipotesis

Ho: $H_0:\sigma_a^2=0$ lawan $H_1:\sigma_a^2>0$

Anova satu arah untuk model efek random

Sumber Variasi	db	Sum of Squares	MS	Expected Mean Squares
Grup	$I-1$	$SSA=J\sum_{i=1}^{I}\left(\overline{y}_{i.}-\overline{y}_{..}\right)^2$	SSA/(*I*-*1*)	$J\sigma_a^2+\sigma^2$
Error	$IJ-I$	$SSE=\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\overline{y}_{i.}\right)^2$	SSE/(*IJ*-1)	σ^2
Total	$IJ-1$	$SST=\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\overline{y}_{..}\right)^2$		

Jika H_0 benar, maka E(MSA) = E(MSE). Dengan demikian, tolak H_0 jika

$$F_{Hit}=\frac{MSA}{MSE}>F_{\alpha;(I-1),(IJ-I)} \tag{1.7}$$